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A PRECISE DEFINITION OF SEPARATION OF VARIABLES

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A precise definition of separation of variables*)

by

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ABSTRACT

We give a precise and conceptual definition of separation of variables for partial differential equations. We derive necessary and sufficient conditions for a linear homogeneous second order partial differential equation to be separable into second order ordinary differential equations. In the case of the Helmholtz equation on a Riemannian manifold these conditions coincide with the classical Stäckel-Robertson conditions. We prove that separability of $Lu + u = 0$ (L second order partial differential operator in n variables) implies the existence of n linearly independent, mutually commuting second order operators, including L. Finally we show that separable second order equations do have an underlying (formal) Riemannian manifold.

KEY WORDS & PHRASES: *separation of variables, Stäckel-Robertson separability conditions, Helmholtz equation on a Riemannian manifold.*

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1. INTRODUCTION

The work presented here originated from a review [8] I wrote on MILLER's [9] book "Symmetry and separation of variables". This book discusses the relationship between group theory and the coordinate systems for which a given partial differential equation is solvable by separation of variables. A remarkable omission in this book is that it does not contain a precise definition for one of the key words: separation of variables. However, when looking in older papers which discuss criteria for separation of variables (cf. ROBERTSON [14], EISENHART [3], MOON & SPENCER [10]), I could not find a precise definition either. Probably the applied mathematician will not be bothered very much by this omission, since he will have a fairly good informal notion of the method of separation of variables, which he can use in ad hoc cases. But if one wants to prove general theorems giving necessary and sufficient conditions for separation of variables or if one wants to classify all separable coordinates for a given partial differential equation then it becomes crucial to have a precise definition.

In section 2, after discussing some definitions from literature, I will propose a new definition of separation of variables which meets the three requirements of (i) being precise, (ii) being conceptual rather than formal, (iii) admitting a rigorous proof that the Stäckel-Robertson conditions (cf. ROBERTSON [14]) are necessary and sufficient for separability of the Helmholtz equation on a Riemannian manifold in given coordinates. In section 3 I derive Stäckel-Robertson type conditions which are necessary and sufficient for a linear homogeneous second order partial differential equation to separate into second order ordinary differential equations. Without proof I state some generalizations of this result for higher order equations and for non-linear equations. As a side result I show that a certain transformed version of the two-variable sine-Gordon equation, which is known to separate into first order o.d.e.'s, cannot separate into second order o.d.e.'s.

Section 4 deals with certain conditions equivalent to Stäckel's condition. The main result relates separability of $Lu = 0$ (L linear second order operator) with a family of $n-1$ linearly independent, mutually commuting second order operators commuting with ΦL for some function Φ. In section 5 we will see that it is no accident that classical and recent work on separation of variables is focused on equations living on (pseudo-) Riemannian manifolds: A general separable linear second order equation has an underlying (formal) Riemannian manifold. The paper concludes with a result stating that on an Einstein manifold the separability of $\Delta u + C(x)u = 0$ for some function C will imply the separability of $\Delta u + u = 0$.

2. THE DEFINITION OF SEPARATION OF VARIABLES

Let us start with a simple example. The Helmholtz equation in two variables

$$u_{xx} + u_{yy} + \omega^2 u = 0$$

is certainly separable as it stands. A slightly less trivial case occurs when we introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + \omega^2 u = 0. \tag{2.1}$$

Now suppose u is a function of the form $u(r,\theta) = f(r)g(\theta)$, not identically zero. Then u is a solution of (2.1) if and only if there is some constant k such that f and g satisfy the ordinary differential equations

$$\begin{cases} r^2 f''(r) + rf'(r) + (\omega^2 r^2 - k^2)f(r) = 0 \\ g''(\theta) + k^2 g(\theta) = 0. \end{cases} \tag{2.2}$$

The general solutions of the o.d.e.'s (2.2) are

$$\begin{cases} f(r) = \alpha_1 J_k(\omega r) + \alpha_2 J_{-k}(\omega r), \\ g(\theta) = \beta_1 e^{ik\theta} + \beta_2 e^{-ik\theta}, \end{cases}$$

where J_k and J_{-k} are Bessel functions.

This example illustrates the way special functions arose in history: as factorized solutions of the p.d.e.'s of mathematical physics when written in separable coordinates. Suitable boundary conditions for (2.1) will restrict the generality of the solutions to be considered for (2.2). More general solutions of (2.1) can be obtained as linear combinations of factorized solutions. However, all these aspects will not bother us in this paper. We will concentrate on making precise the relationship between a p.d.e. like (2.1) and o.d.e.'s like (2.2).

Historically, a more systematic research on separability for p.d.e.'s was done in the context of a Riemannian manifold. For local coordinates $x_1, x_2, \ldots, x_n$ on the manifold let $g_{ij}(x)$ be the fundamental tensor and write $g_i := g_{ii}$. Assume the coordinates are orthogonal, i.e., the tensor g_{ij} is diagonal. STÄCKEL [16] proved in 1891 that the Hamilton-Jacobi equation

$$(2.3)\qquad \sum_{i=1}^{n} g_i^{-1}\left(\frac{\partial u}{\partial x_i}\right)^2 = \omega^2 \qquad (\omega>0)$$

is separable by solutions of the form $u(x) = X_1(x_1) +\ldots+ X_n(x_n)$ if and only if the following condition (the so-called Stäckel condition) holds:

(i) There is a nonsingular $n \times n$ matrix $(c_{ij}(x_i))$ with inverse $(\gamma_{ij}(x))$ such that $(g_i(x))^{-1} = \gamma_{1i}(x)$.

For orthogonal local coordinates $(x_1,\ldots,x_n)$ the Laplace-Beltrami operator Δ on the Riemannian manifold takes the form

$$(2.4)\qquad \Delta = \sum_{i=1}^{n} g^{-\frac{1}{2}} \frac{\partial}{\partial x_i} \circ g^{\frac{1}{2}} g_i^{-\frac{1}{2}} \frac{\partial}{\partial x_i},$$

where $g := \det(g_{ij}) = \Pi_{i=1}^{n} g_i$. ROBERTSON [14] proved in 1928 that the Helmholtz type equation

$$(2.5)\qquad \Delta u + \omega^2 u = 0 \qquad (\omega>0)$$

is separable by solutions of the form $u(x) = \Pi_{i=1}^{n} X_i(x_i)$ if and only if the Stäckel condition (i) holds and, furthermore:

(ii) There are functions f_k $(k = 1,\ldots,n)$ in one variable such that

$$\frac{(g(x))^{\frac{1}{2}}}{\det(c_{ij}(x_i))} = \prod_{k=1}^{n} f_k(x_k).$$

EISENHART [3] (see also [4, Appendix 13]) observed in 1934 that Robertson's condition (ii) can be replaced by the condition (cf. section 5):

(ii)' The Ricci tensor $R_{ij}(x)$ is diagonal.

For certain classes of Riemannian manifolds, for instance for flat spaces, this condition is always satisfied if g_{ij} is diagonal. By way of application, EISENHART [3] classified the eleven separable coordinate systems for the three-variable Helmholtz equation

$$(2.6)\qquad u_{xx} + u_{yy} + u_{zz} + \omega^2 u = 0.$$

Although the above-mentioned authors state and prove necessary and sufficient conditions for the separability of (2.3) or (2.5), they do not give a precise definition of separation of variables. Therefore, in order to learn about the definition they have in mind, we have to look for the implicit assumptions they make in their proofs. Let us consider ROBERTSON [14] (with his potential energy V being zero), see also the same proof in MOON & SPENCER [10, Theorem 1]. In the necessity proof of conditions (i) and (ii) Robertson states that separability of (2.5) implies that the coefficients of $\partial^2u/\partial x_i^2$ and $\partial u/\partial x_i$ in (2.5) are proportional by a factor only depending on x_i. Furthermore, he concludes from the separability assumption that there must be a family of factorized solutions $u(x) = \Pi_{i=1}^n X_i(x_i)$ of (2.5) depending on n parameters $\alpha_1 := \omega^2, \alpha_2,\ldots,\alpha_n$ such that

$$\det\left(\frac{\partial}{\partial\alpha_j}[X_i^{-1}(f_i(x_i)X_i'(x_i))']\right) \neq 0$$

(f_i as in condition (ii)) for some $(\alpha_1,\ldots,\alpha_n)$. Apparently, Robertson assumes that (2.5) is simultaneously separable for all values of ω. On the other hand, Robertson proves the sufficiency of conditions (i), (ii) by showing that there are n o.d.e.'s jointly depending on parameters $\alpha_1 := \omega^2,\alpha_2,\ldots,\alpha_n$ such that, if $u(x) = \Pi_{i=1}^n X_i(x_i)$ satisfies (2.5), then $X_1,\ldots,X_n$ satisfy the o.d.e.'s for some choice of $(\alpha_1,\ldots,\alpha_n)$. I think it is difficult to extract from the above elements a clear and precise picture of the definition of separation of variables Robertson had in mind.

Let me next discuss three different definitions of separation of variables I met in literature.

DEFINITION A. Cf. MORSE & FESHBACH [11, §5.1, p.497].
Write the three-variable Helmholtz equation (2.6) in new coordinates ξ_1, ξ_2, ξ_3. These coordinates are called *separable* for equation (2.6) if each solution of (2.6) is a linear combination of solutions of the form $F_1(\xi_1)F_2(\xi_2)F_3(\xi_3)$.

Discussion. This is typically a definition from the user's point of view: one can use the method of separation of variables if one can build the general solution as a linear combination of factorized solutions. However, the definition is not precise, since it is not specified in which topology these linear combinations have to converge. It will also be hard to derive a separability criterium or a classification result starting from this definition. Finally it is remarkable that the definition does not require that the functions F_i satisfy certain o.d.e.'s.

DEFINITION B. Cf. SNEDDON [15, Ch.3, §9, p.123].
A second order homogeneous linear p.d.e. in two variables

$$A_{11}u_{xx} + 2A_{12}u_{xy} + A_{22}u_{yy} + B_1u_x + B_2u_y + Cu = 0 \tag{2.7}$$

is called *separable* if, for each solution u of the form $u(x,y) = X(x)Y(y)$, equation (2.7) can be written as

$$X^{-1}D_1X = Y^{-1}D_2Y, \tag{2.8}$$

where D_1 and D_2 are second order ordinary differential operators in x and y, respectively.

Discussion. This is a formal criterium for separability. By manipulation of (2.7) one has to try to achieve an equation of the form (2.8). It is apparent that (2.8) separates into two o.d.e.'s, but it is left unspecified what is meant by this last and most important step.

DEFINITION C. Cf. NIESSEN [12, p.329].
A linear partial differential operator L in $x_1,\dots,x_n$ is called *separable* if there is an $n \times n$ matrix (L_{ij}), with the matrix element L_{ij} being an ordinary (possibly zero order) linear differential operator in x_j, such that, for all sufficiently often differentiable functions $x_i \to X_i(x_i)$ $(i = 1,\dots,n)$ we have

$$(2.9) \qquad L\Big(\prod_{i=1}^{n} X_i\Big) = \det(L_{ij}X_j).$$

Discussion. Call the equation $Lu = 0$ separable if the operator L is separable according to Definition C. Definition B can be viewed as a special case of Definition C if we put

$$(L_{ij}X_j) := \begin{pmatrix} D_1X_1 & D_2X_2 \\ X_1 & X_2 \end{pmatrix}.$$

The criticism to Definition B also applies here. Further objections are that the definition does not cover nonlinear equations or the case of a linear second order p.d.e. separating into first order o.d.e.'s. However, positive aspects of Definitions B and C are their preciseness and the fact that they can easily be used for the derivation of separability criteria.

I will conclude this section by proposing yet another definition of separation of variables, which will meet the three requirements mentioned in the introduction.

Let $m \in \mathbb{N}$. Consider for real $x_1,x_2,\dots,x_n$ the p.d.e.

$$(2.10) \qquad F\left(\frac{\partial^{i_1+\dots+i_n}}{\partial x_1^{i_1}\dots x_n^{i_n}}\, u(x_1,\dots,x_n), x_1,\dots,x_n\right) = 0,$$

where the derivatives of u are running over all orders such that $i_1+\dots+i_n \le m$, where u is allowed to be a complex-valued function, and where F is assumed to be analytic in all its arguments.

DEFINITION 2.1. The p.d.e. (2.10) is called *separable* for $(x_1,\dots,x_n)$ lying in some open connected region $\Omega \subset \mathbb{R}^n$ if there are n analytic o.d.e.'s

$$(2.11) \qquad X_i^{(k_i)}(x_i) + f_i(X_i^{(k_i-1)}(x_i),\dots,X_i(x_i),x_i,\alpha_2,\dots,\alpha_n) = 0, \qquad i = 1,\dots,n,$$

jointly depending in an analytic way on n-1 independent complex parameters $\alpha_2,\dots,\alpha_n$, such that, for each $(\alpha_2,\dots,\alpha_n)$ and for each set of solutions $(X_1,\dots,X_n)$ of (2.11) with arguments $(x_1,\dots,x_n)$ in Ω, the function

$$u(x_1,\dots,x_n) := \prod_{i=1}^{n} X_i(x_i) \tag{2.12}$$

is a solution of (2.10).

DEFINITION 2.2. The n-1 complex parameters $\alpha_2,\dots,\alpha_n$ in (2.11) are called *independent* if the $n\times(n-1)$ matrix

$$\left(\frac{\partial f_i}{\partial \alpha_j}(y_{k_i-1,i}, y_{k_i-2,i},\dots,y_{0,i},x_i,\alpha_2,\dots,\alpha_n)\right)$$

has rank n-1 whenever $\prod_{i=1}^{n} y_{0,i} \neq 0$.

REMARK 2.3. If the function F in (2.10) is not defined globally as a function of u and its derivatives then suitable modifications have to be made in Definition 2.1, such that it can be understood locally.

REMARK 2.4. The requirement of analyticity for (2.10) and (2.11) is not very stringent, but just for convenience. Because an analytic function in one variable, not identically zero, has the properties that it is completely determined by its restriction to some real interval and that its zeros are isolated, we will be able in later proofs to divide by such a function, neglecting possible zeros.

REMARK 2.5. In certain circumstances, the condition $\prod_{i=1}^{n} y_{0,i} \neq 0$ in Definition 2.2 may not be the right choice. Anyhow, for each value of $x_1,\dots,x_n$, $\alpha_2,\dots,\alpha_n$, the rank of $(\partial f_i/\partial \alpha_j)$ should be n-1 for generic values of $y_{k_i-1,i},\dots,y_{0,i}$ $(i = 1,\dots,n)$.

REMARK 2.6. Under the terms of Definition 2.1 a converse implication often holds: If u is a function of the form (2.12), analytic and not identically zero, and if u satisfies (2.10) then the functions X_i, $i = 1,\dots,n$, satisfy (2.11) for some choice of the parameters $\alpha_2,\dots,\alpha_n$. In section 3 we will prove this converse implication in the case of a linear second order p.d.e. which separates into second order o.d.e.'s.

REMARK 2.7. It is easy to make a connection between our Definition 2.1 and NIESSEN's [12] Definition C. Let the linear partial differential operator L have the property of formula (2.9). Now, for each value of $(\alpha_1,\alpha_2,\dots,\alpha_n)$, if the functions $X_1,\dots,X_n$ satisfy the o.d.e.'s

$$\sum_{i=1}^{n} \alpha_i L_{ij} X_j(x_j) = 0$$

then $u := \prod_{i=1}^{n} X_i$ satisfies $Lu = 0$. However, without further assumptions on the L_{ij}'s it is not clear whether n-1 of the parameters $\alpha_1,\dots,\alpha_n$ form a set of independent parameters for this set of o.d.e.'s.

EXAMPLE 2.8. Clearly the p.d.e. (2.1) separates into the o.d.e.'s (2.2) according to Definition 2.1 and also the converse implication of Remark 2.6 holds. Similarly, the p.d.e.

$$u_{xx} + u_{yy} + \omega^2 u = 0 \tag{2.13}$$

separates, under the Ansatz $u(x,y) = X(x)Y(y)$, into the o.d.e.'s

$$\begin{cases} X''(x) + (\omega^2-k^2)X(x) = 0, \\ Y''(y) + k^2 Y(y) = 0, \end{cases}$$

and again the converse implication holds. However note that (2.13) also separates into the first order o.d.e.'s

$$\begin{cases} X'(x) + i\sqrt{\omega^2-k^2}\, X(x) = 0, \\ Y'(y) + ikY(y) = 0. \end{cases} \tag{2.14}$$

The converse implication of Remark 2.6 no longer holds in this case, but it does hold for a three-parameter family of pairs of o.d.e.'s extending (2.14): Let u be a function of the form $u(x,y) = X(x)Y(y)$, not identically zero. Then u satisfies (2.13) if and only if X and Y satisfy the o.d.e's

$$\begin{cases} (X'(x))^2 + (\omega^2-k^2)(X(x))^2 = A^2, \\ (Y'(y))^2 + k^2(Y(y))^2 = B^2 \end{cases}$$

for some value of (k,A,B).

EXAMPLE 2.9. Consider the sine-Gordon equation

$$\phi_{xx} - \phi_{tt} = \sin\phi.$$

Put $u(x,t) := tg(\frac{1}{4}\phi(x,t))$. The transformed equation reads

$$(1+u^2)(u_{xx}-u_{tt}) - 2u(u_x^2-u_t^2) = u(1-u^2). \tag{2.15}$$

Under the Ansatz $u(x,t) = X(x)T(t)$ this nonlinear p.d.e. separates into the first order o.d.e.'s

$$\begin{cases} X'(x) = \alpha^{\frac{1}{2}}X(x) \\ T'(t) = \sqrt{\alpha-1}\; T(t), \end{cases}$$

according to Definition 2.1. On the other hand, let $u(x,t) = X(x)T(t)$ be an analytic solution of (2.15) such that u_x and u_t are not identically zero. Then it can be shown that, for some choice of α,β,γ, the functions X and T satisfy the o.d.e.'s

$$\begin{cases} (X')^2 = \beta X^4 + \alpha X^2 + \gamma, \\ (T')^2 = -\gamma T^4 + (\alpha-1)T^2 - \beta, \end{cases}$$

see also OSBORNE & STUART [13]. It will follow from Lemma 3.6 that equation (2.15) does not separate into second order o.d.e.'s.

3. SEPARATION OF VARIABLES FOR LINEAR SECOND ORDER EQUATIONS

In this section we will derive criteria for separability of a general linear homogeneous second order p.d.e.

$$(3.1) \qquad \sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i(x) \frac{\partial u}{\partial x_i} + C(x)u = 0.$$

Here A_{ij}, B and C are complex-valued analytic functions of $x = (x_1,\dots,x_n)$ on some open connected region Ω in $\mathbb{R}^n$. We may assume $A_{ij} = A_{ji}$ and we will write $A_i := A_{ii}$. Furthermore, we require that, for each i, (3.1) contains some nonvanishing term involving a derivative with respect to x_i, i.e., for each $x \in \Omega$ and for each $i \in \{1,\dots,n\}$ not all of the numbers $A_{i1}(x), A_{i2}(x),\dots,A_{in}(x), B_i(x)$ are zero. Let us formulate the main theorem.

<u>THEOREM 3.1</u>. *The p.d.e.* (3.1) *separates on* Ω *into* n *second order o.d.e.'s if and only if the following three conditions hold*:

(i) $A_{ij} = 0$ *if* $i \neq j$ *and* $A_i(x) \neq 0$ *for all* x *and* i.

(ii) *There are analytic functions* b_i $(i = 1,\dots,n)$ *in one real variable such that*

$$(3.2) \qquad B_i(x) = b_i(x_i)A_i(x).$$

(iii) *There are analytic functions* c_{ij} $(i,j = 1,\dots,n)$ *in one real variable such that the* $n \times (n-1)$ *matrix*

$$(3.3) \qquad \begin{pmatrix} c_{12}(x_1)\dots c_{1n}(x_1) \\ c_{n2}(x_n)\dots c_{nn}(x_n) \end{pmatrix}$$

has rank $n-1$ *for all* $x = (x_1,\dots,x_n) \in \Omega$ *and*

$$(3.4) \qquad \sum_{i=1}^{n} c_{ij}(x_i)A_i(x) = C(x)\delta_{j,1}, \; j = 1,\dots,n.$$

Under the assumption of conditions (i), (ii), (iii), *the p.d.e.* (3.11) *takes the form*

$$(3.5) \qquad \sum_{i=1}^{n} A_i(x)\left(\frac{\partial^2 u}{\partial x_i^2} + b_i(x_i)\frac{\partial u}{\partial x_i} + c_{i1}(x_i)u\right) = 0.$$

Furthermore, if a function u *has the form* $u(x) = \prod_{i=1}^{n} X_i(x_i)$ *and if* u *is not identically zero on* Ω, *then* u *is a solution of* (3.5) *if and only if, for some* $(\alpha_2,\dots,\alpha_n) \in \mathbb{C}^{n-1}$, *the functions* X_i *are solutions of the o.d.e.'s*

$$(3.6) \qquad X_i''(x_i)+b_i(x_i)X_i'(x_i)+\left(c_{i1}(x_i)+\sum_{j=2}^{n} c_{ij}(x_i)\right)X_i(x_i) = 0, \qquad i = 1,\dots,n.$$

For the proof we will need a lemma, see Lemma 3.2. First we introduce some notation and we formulate alternatives to condition (iii). Consider an $(n\times n)$ matrix-valued function

$$(3.7) \qquad c(x) := (c_{ij}(x_i)).$$

Let

$$(3.8) \qquad M_{ij}(x) := \text{cofactor of } c(x) \text{ for entry}(i,j),$$

$$(3.9) \qquad S(x) := \det c(x).$$

$M_{ij}(x)$ only depends on $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$. If conditions (i) and (iii) of the theorem hold then, by (3.4),

$$(3.10) \qquad \frac{A_i(x)}{A_j(x)} = \frac{M_{i1}(x)}{M_{j1}(x)}, \qquad i,j = 1,\dots,n,$$

and, hence, $M_{i1}(x) \neq 0$ for all x and i.

Let condition (i) hold. If $C(x) = 0$ for all x then (iii) is equivalent to:

(iii)' *There is an analytic matrix-valued function* c *of the form* (3.7) *such that, for all* $x \in \Omega$, *the matrix* (3.3) *has rank* $n-1$, $c_{i1}(x_i) = 0$ $(i = 1,\dots,n)$ *and* (3.10) *holds.*

If $C(x) \neq 0$ for all $x \in \Omega$ then (iii) is equivalent to:

(iii)" *There is an analytic matrix-valued function* c *of the form* (3.7) *such that, for all* $x \in \Omega$, $S(x) \neq 0$ *and*

$$(3.11) \qquad A_i(x) = \frac{C(x)M_{i1}(x)}{S(x)}.$$

Note that (3.11) can also be written as $A_i(x) = C(x)(c(x)^{-1})_{1i}$, where $c(x)^{-1}$ is the matrix inverse of $c(x)$.

LEMMA 3.2. *Suppose there are* n *second order o.d.e.'s*

$$(3.12)\qquad X_i''(x_i) + f_i(X_i'(x_i),\ X_i(x_i), x_i) = 0, \qquad i = 1,\ldots,n,$$

with f_i *analytic, such that for each set of solutions* $(X_1,\ldots,X_n)$ *of* (3.12) *the function* u *given by* $u(x) = \prod_{i=1}^n X_i(x_i)$ *is a solution of* (3.1). *Then condition* (i) *of Theorem* 3.1 *holds and there are analytic functions* b_i *and* c_i $(i = 1,\ldots,n)$ *in one real variable such that* (3.2) *is valid and also*

$$(3.13)\qquad C(x) = \sum_{i=1}^{n} c_i(x_i)A_i(x),$$

$$(3.14)\qquad f_i(X_i'(x_i),X_i(x_i),x_i) = b_i(x_i)X_i'(x_i)+c_i(x_i)X_i(x_i), \qquad i = 1,\ldots,n.$$

PROOF. Substitute (3.12) into (3.1) with $u = \prod_{i=1}^n X_i$. Then

$$(3.15)\qquad -\sum_{i=1}^{n} A_i(x)\,\frac{f_i(X_i'(x_i),X_i(x_i),x_i)}{X_i(x_i)} + \sum_{i\neq j} A_{ij}(x)\,\frac{X_i'(x_i)X_j'(x_j)}{X_i(x_i)X_j(x_j)} + \sum_{i=1}^{n} B_i(x)\,\frac{X_i'(x_i)}{X_i(x_i)} + C(x) = 0.$$

It follows from the assumptions in the lemma and from the theory of second order ordinary differential equations that, for each $x \in \Omega$, equation (3.15) will be satisfied for all complex values of $X_i'(x_i)$ $(i = 1,\ldots,n)$ and for all nonzero complex values of $X_i(x_i)$ $(i = 1,\ldots,n)$. Fix $x \in \Omega$. Successive differentiation of (3.15) with respect to $X_i'(x_i)$ and $X_j'(x_j)$ $(i \neq j)$ yields that $A_{ij}(x) = 0$ for $i \neq j$. Next, by differentiation of (3.15) with respect to $X_i'(x_i)$ we obtain

$$B_i(x) = A_i(x)\,\frac{\partial f_i(X_i'(x_i),X_i(x_i),x_i)}{\partial X_i'(x_i)}\,.$$

$A_i(x) = 0$ would imply $B_i(x) = 0$, contradicting the original assumptions about (3.1). So $A_i(x) \neq 0$ and condition (i) of Theorem 3.1 is proved. It also follows that

$$(3.16)\qquad f_i(X_i'(x_i),X_i(x_i),x_i) = b_i(x_i)X_i'(x_i)+c_i(X_i(x_i),x_i)X_i(x_i)$$

for certain analytic functions b_i and c_i, with b_i satisfying (3.2). Substitution of (3.16) and (3.2) into (3.15) yields

$$C(x) = \sum_{i=1}^{n} c_i(X_i(x_i),x_i)A_i(x).$$

By differentiating this formula with respect to $X_i(x_i)$ we obtain

$\partial c_i(X_i(x_i),x_i)/\partial X_i(x_i) = 0$. Thus c_i only depends on x_i. Now (3.13) and (3.14) are proved. □

Lemma 3.2 states that, if a linear homogeneous second order p.d.e. (3.1) "separates" into one set of n o.d.e.'s (3.12), not a priori linear and not depending on additional parameters, then this assumption already forces the o.d.e.'s to be linear and also imposes strong restrictions on the coefficients A_i, B_i, C in (3.1). However, for the proof of this lemma it seems to be crucial to assume that $u = \prod_{i=1}^{n} X_i$ satisfies (3.1) for all possible solutions $(X_1,\ldots,X_n)$ of the o.d.e.'s, not just for one set of solutions.

PROOF of Theorem 3.1.

(a) *Necessity of the conditions* (i), (ii), (iii).
Suppose (3.1) separates into n second order o.d.e.'s. Then we know from Lemma 3.2 that conditions (i) and (ii) are satisfied and that (3.1) separates into o.d.e.'s of the form

$$X_i''(x_i)+b_i(x_i)X_i'(x_i)+c_i(x_i,\alpha_2,\ldots,\alpha_n)X_i(x_i) = 0, \qquad i = 1,\ldots,n,$$

where b_i satisfies (3.2) and

$$\sum_{i=1}^{n} c_i(x_i,\alpha_2,\ldots,\alpha_n)A_i(x) = C(x).$$

Differentiate the last identity with respect to α_j:

$$\sum_{i=1}^{n} \frac{\partial c_i}{\partial \alpha_j}(x_i,\alpha_2,\ldots,\alpha_n)A_i(x) = 0, \qquad j = 2,\ldots,n.$$

By Definitions 2.1 and 2.2 the $n \times (n-1)$ matrix $\left(\frac{\partial c_i}{\partial \alpha_j}(x_i,\alpha_2,\ldots,\alpha_n)\right)$ has rank n-1. Choose a fixed $(\alpha_2,\ldots,\alpha_n)$ and define

$$c_{i1}(x_i) := c_i(x_i,\alpha_2,\ldots,\alpha_n), \qquad i = 1,\ldots,n,$$

$$c_{ij}(x_i) := \frac{\partial c_i}{\partial \alpha_j}(x_i,\alpha_2,\ldots,\alpha_n), \qquad i = 1,\ldots,n,\ j = 2,\ldots,n.$$

With this choice of the functions c_{ij} condition (iii) holds.
(b) *Sufficiency of the conditions* (i), (ii), (iii).
Assume conditions (i), (ii), (iii) hold. Then (3.1) takes the form (3.5). Let, for some $(\alpha_2,\ldots,\alpha_n) \in \mathbb{C}^{n-1}$, $(X_1,\ldots,X_n)$ be a set of solutions of (3.6), with $X_i(x_i) \neq 0$ for all x and i. Multiply (3.6) by $A_i(x)/X_i(x_i)$ and sum over i. Using (3.4) we obtain

$$(3.17) \qquad \sum_{i=1}^{n} A_i(x)\left(\frac{X_i''(x_i)}{X_i(x_i)} + b_i(x_i)\,\frac{X_i'(x_i)}{X_i(x_i)} + c_{i1}(x_i)\right) = 0.$$

If $u = \prod_{i=1}^{n} X_i$ then (3.17) implies (3.5).

(c) *Proof that the factors in the factorized solutions of* (3.5) *satisfy* (3.6). Without loss of generality we may assume $X_i(x_i) \neq 0$ (cf. Remark 2.4). Compare (3.17) with the n-1 equations (3.4) for $j = 2,\ldots,n$. Since the matrix (3.3) has rank n-1 we conclude that

$$\frac{X_i''(x_i)}{X_i(x_i)} + b_i(x_i)\,\frac{X_i'(x_i)}{X_i(x_i)} + c_{i1}(x_i) + \sum_{j=2}^{n} \alpha_j(x)c_{ij}(x_i) = 0$$

for certain coefficients α_j $(j = 1,\ldots,n)$ depending on x. Hence, for fixed ℓ:

$$\sum_{j=2}^{n} (\alpha_j(x_1,\ldots,\tilde{x}_\ell,\ldots,x_n)-\alpha_j(x_1,\ldots,x_\ell,\ldots,x_n))\; c_{ij}(x_i) = 0,$$

$$i = 1,\ldots,\ell-1,\ell+1,\ldots,n.$$

The determinant of the $(n-1)\times(n-1)$ matrix which is obtained by deleting the ℓ^{th} row and the first column in $(c_{ij}(x_i))$, equals $(-1)^{\ell+1}\, M_{\ell 1}(x)$ (cf. (3.8)). Because of (3.10) this determinant is nonzero. Hence $\alpha_j(x_1,\ldots,\tilde{x}_\ell,\ldots,x_n) = \alpha_j(x_1,\ldots,x_\ell,\ldots,x_n)$, so α_j does not depend on x_ℓ $(\ell = 1,\ldots,n)$. Thus (3.6) holds for these α_j. □

REMARK 3.3. The relationship between our Theorem 3.1 and NIESSEN's [12] definition C is easily established (cf. also Remark 2.7). Indeed, the p.d.e. (3.1) satisfies the conditions (i), (ii), (iii) of Theorem 3.1 if and only if the left hand side of (3.1), after multiplication by some function of x and for functions u of the form $u(x) = \prod_{i=1}^{n} X_i(x_i)$, can be written as $\det(L_{ij}\, X_j(x_j))$, where

$$L_{1j} := \frac{d^2}{dx_j^2} + b_j(x_j)\,\frac{d}{dx_j} + c_{j1}(x_j),$$

$$L_{ij} := c_{ji}(x_j), \qquad i \neq 1,$$

and the matrix (3.3) has rank n-1.

REMARK 3.4. Consider the Helmholtz equation (2.5) on a Riemannian manifold. On comparing (2.5) with (3.1) we have:

$$A_i = \frac{1}{g_i}\;,\; A_{ij} = 0 \;\text{ if } i \neq j, \qquad C = \omega^2,$$

$$B_i = \frac{1}{g_i}\,\frac{\partial}{\partial x_i}\,(\log \frac{g^{\frac{1}{2}}}{g_i}).$$

If A_i takes the form (3.11) then

$$\frac{B_i}{A_i} = \frac{\partial}{\partial x_i}\,\log\,(\frac{g^{\frac{1}{2}}}{S}).$$

Hence our separability conditions (i), (ii), (iii) of Theorem 3.1, if applied to the p.d.e. (2.5), are equivalent to the Stäckel-Robertson conditions (i) and (ii), mentioned in §2.

Next we mention some generalizations of Theorem 3.1 and Lemma 3.2 to higher order and nonlinear p.d.e.'s. The proofs, which are omitted, are similar to the proofs earlier in this section.

LEMMA 3.5. *Consider a linear homogeneous analytic p.d.e. of* m^{th} *order*

$$\sum_{|\alpha|\le m} A_\alpha(x) \frac{\partial^{\alpha_1+\dots+\alpha_n} u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}} = 0 \tag{3.18}$$

such that, for each x *and* i, *not all terms involving derivatives with respect to* x_i *will vanish. Suppose there are* n *analytic o.d.e.'s of* m^{th} *order*

$$X_i^{(m)}(x_i)+f_i(X_i^{(m-1)}(x_i),\dots,X_i(x_i),x_i) = 0, \qquad i = 1,\dots,n, \tag{3.19}$$

such that, for each set of solutions $(X_1,\dots,X_n)$ *of* (3.19), *the function* $u(x) := \prod_{i=1}^n X_i(x_i)$ *is a solution of* (3.18). *Then* (3.18) *and* (3.19) *must have the form*

$$\sum_{i=1}^n A_i(x)\left(\frac{\partial^m u}{\partial x_i^m} + \sum_{j=1}^{m-1} b_i^j(x_i) \frac{\partial^j u}{\partial x_i^j}\right) + C(x)u = 0,$$

$$X_i^{(m)}(x_i) + \sum_{j=1}^{m-1} b_i^j(x_i)X_i^{(j)}(x_i)+c_i(x_i)X_i(x_i) = 0, \qquad i = 1,\dots,n,$$

respectively, for certain analytic functions b_i^j, c_i, *where*

$$C(x) = \sum_{i=1}^n A_i(x)c_i(x_i)$$

and $A_i(x) \neq 0$ *for all* x *and* i.

LEMMA 3.6. *Consider a (generally nonlinear) second order p.d.e. of the form*

$$\sum_{i=1}^n A_i(x) \frac{\partial^2 u}{\partial x_i^2} + F\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},u,x\right) = 0, \tag{3.20}$$

with A_i *and* F *analytic and* $A_i(x) \neq 0$ *for all* x *and* i. *Suppose there are* n *analytic second order o.d.e.'s*

$$X_i''(x_i)+f_i(X_i'(x_i),X_i(x_i),x_i) = 0, \qquad i = 1,\dots,n, \tag{3.21}$$

such that, for each set of solutions $(X_1,\dots,X_n)$ *of* (3.21), *the function* $u(x) := = \prod_{i=1}^n X_i(x_i)$ *is a solution of* (3.22). *Then* (3.20) *and* (3.21) *must have the form*

$$\sum_{i=1}^{n} A_i(x)\left(\frac{\partial^2 u}{\partial x_i^2} + \beta_i\left(\frac{\partial u}{\partial x_i}, u, x_i\right)\right) + D(x)\, u \log u = 0,$$

$$X_i''(x_i) + \beta_i(X_i'(x_i), X_i(x_i), x_i) + d_i(x_i) X_i(x_i) \log X_i(x_i) = 0, \quad i = 1,\ldots,n,$$

respectively, for certain analytic β_i, d_i *and* D, *where* β_i *is homogeneous of degree* 1 *in its first two arguments and*

(3.22) $$A_i(x) d_i(x_i) = D(x), \quad i = 1,\ldots,n.$$

Note that, in case D is not identically zero, equation (3.22) highly restricts the possible choices for the A_i's.

REMARK 3.7. It follows from Lemma 3.6 that the nonlinear second order p.d.e. (2.15) does not separate into second order o.d.e.'s.

THEOREM 3.8. *Consider a (generally nonlinear)* m^{th} *order p.d.e.*

(3.23) $$\sum_{i=1}^{n} A_i(x)\left(\frac{\partial^m u}{\partial x_i^m} + \beta_i\left(\frac{\partial^{m-1} u}{\partial x_i^{m-1}}, \ldots, \frac{\partial u}{\partial x_i}, u, x_i\right)\right) = 0,$$

where A_i *and* β_i *are analytic,* $A_i(x) \neq 0$ *for all* x *and* i, *and* β_i *is homogeneous of degree* 1 *in* $\partial^{m-1}u/\partial x_i^{m-1},\ldots,\partial u/\partial x_i, u$. *Then* (3.23) *separates into* n m^{th} *order o.d.e.'s if and only if there are analytic functions* c_{ij} (i = 1,...,n, j = 2,...,n) *in one real variable such that the* n × (n-1) *matrix* $(c_{ij}(x_i))$ *has rank* n-1 *for all* x *and*

$$\sum_{i=1}^{n} c_{ij}(x_i) A_i(x) = 0, \quad j = 2,\ldots,n.$$

In case of separability a function $u(x) = \prod_{i=1}^{n} X_i(x_i)$, *not identically zero, is a solution of* (3.23) *if and only if, for some* $(\alpha_2,\ldots,\alpha_n) \in \mathbb{C}^{n-1}$, *the functions* X_i *are solutions of the o.d.e.'s*

$$X_i^{(m)}(x_i) + \beta_i(X_i^{(m-1)}(x_i),\ldots,X_i'(x_i),X_i(x_i),x_i) + $$
$$+ \left(\sum_{j=2}^{n} \alpha_j\, c_{ij}(x_i)\right) X_i(x_i) = 0, \quad i = 1,\ldots,n.$$

Finally we turn to the case of the Hamilton-Jacobi equation (2.3) considered by STÄCKEL [16]. We will now use Definition 2.1 with the Ansatz (2.12) replaced by

(3.24) $$u(x) = \sum_{i=1}^{n} X_i(x_i).$$

In Definition 2.2 the condition $\prod_{i=1}^{n} y_{0,i} \neq 0$ can then be omitted. It is easy to prove the following analogue of Theorem 3.1.

THEOREM 3.9. *Consider the first order nonlinear p.d.e.*

$$\sum_{i=1}^{n} A_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2 = C(x), \tag{3.25}$$

where A_i *and* C *are analytic and* $A_i(x) \neq 0$ *for all* x *and* i. *Then the p.d.e.* (3.25) *separates into first order o.d.e.'s under the Ansatz* (3.24) *if and only if condition* (iii) *of Theorem* 3.1 *holds.*

4. ON CONDITIONS EQUIVALENT TO STACKEL's CRITERIUM

The main result in this section is Theorem 4.5, which associates n linearly independent, mutually commuting partial differential operators with a separable second order p.d.e.. Theorem 4.6 and Corollary 4.7 will be needed in section 5.

Lemma 4.2 and Theorems 4.3 and 4.4 will involve:

ASSUMPTION 4.1. The functions c_{ij} $(i,j = 1,\dots,n)$ are analytic on an open connected region $\Omega \subset \mathbb{R}^n$ such that $\det(c_{ij}(x)) \neq 0$ on Ω. The matrix inverse of $(c_{ij}(x))$ is denoted by $(\gamma_{ij}(x))$. Assume that $\gamma_{1i}(x) \neq 0$ $(i = 1,\dots,n)$ on Ω.

LEMMA 4.2. *Let* c_{ij} *and* γ_{ij} *be as in Assumption* 4.1. *Then the following three statements are equivalent:*

(a) c_{ij} *only depends on* x_i $(i,j = 1,\dots,n)$.

(b) $\gamma_{kp} \dfrac{\partial \gamma_{1j}}{\partial x_p} = \gamma_{1p} \dfrac{\partial \gamma_{kj}}{\partial x_p}$ $(k = 2,\dots,n;\ j,p = 1,\dots,n)$.

(c) $\gamma_{kp} \dfrac{\partial^m \gamma_{\ell j}}{\partial x_p^m} = \gamma_{\ell p} \dfrac{\partial^m \gamma_{kj}}{\partial x_p^m}$ $(j,k,\ell,p = 1,\dots,n;\ m = 1,2,3,\dots)$.

PROOF. We prove (a) ⇒ (c) ⇒ (b) ⇒ (a).

(a) ⇒ (c): $\sum_k c_{ik}(x_i)\gamma_{kj}(x) = \delta_{ij}$.

Hence

$$\sum_k c_{ik}(x_i) \frac{\partial^m \gamma_{kj}(x)}{\partial x_p^m} = 0, \qquad i \neq p.$$

So

$$\sum_k c_{ik}(x_i)\left(\gamma_{kp} \frac{\partial^m \gamma_{\ell j}}{\partial x_p^m} - \gamma_{\ell p} \frac{\partial^m \gamma_{kj}}{\partial x_p^m}\right) = 0, \qquad i \neq p.$$

Since also

$$\sum_k c_{ik}(x_i)\gamma_{kp}(x) = 0, \qquad i \neq p,$$

it follows that

$$\frac{1}{\gamma_{kp}}\left(\gamma_{kp}\frac{\partial^m\gamma_{\ell j}}{\partial x_p^m}-\gamma_{\ell p}\frac{\partial^m\gamma_{kj}}{\partial x_p^m}\right)$$

is independent of k. So (c) will follow if $\gamma_{\ell p}\neq 0$. Suppose $\gamma_{\ell p}(y) = 0$ for some $y\in\Omega$. Then $\ell \neq 1$. For $\alpha \in \mathbb{C}$ let

$$\tilde{\gamma}_{ij} := \gamma_{ij} + \alpha\delta_{i\ell}\gamma_{1j},$$

$$\tilde{c}_{ij} := c_{ij} - \alpha\delta_{j1}c_{i\ell}.$$

Then $(\tilde{\gamma}_{ij}(x))$ and $(\tilde{c}_{ij}(x_i))$ are matrix inverses of each other and $\tilde{\gamma}_{\ell p}(y) \neq 0$ if $\alpha \neq 0$. Hence (c) is valid for $(\tilde{\gamma}_{ij}(y))$ if $\alpha \neq 0$ and the case $\alpha = 0$ follows by continuity.

(b) ⇒ (a): Let $p \neq i$. Differentiation of

$$\sum_k c_{ik}\gamma_{kj} = \delta_{ij}$$

yields

$$\sum_k \frac{\partial c_{ik}}{\partial x_p}\gamma_{kj} = -\sum_k c_{ik}\frac{\partial\gamma_{kj}}{\partial x_p} = -\frac{1}{\gamma_{1p}}\frac{\partial\gamma_{1j}}{\partial x_p}\sum_k c_{ik}\gamma_{kp} = 0. \qquad \square$$

THEOREM 4.3. *Let c_{ij} and γ_{ij} be as in Assumption 4.1. Let b_i, c_i $(i = 1,\dots,n)$ be analytic functions in one real variable. Consider the n linearly independent partial differential operators*

$$L_k := \sum_{i=1}^n \gamma_{ki}(x)\left(\frac{\partial^2}{\partial x_i^2} + b_i(x_i)\frac{\partial}{\partial x_i} + c_i(x_i)\right), \qquad k = 1,\dots,n.$$

Then the following three statements are equivalent:

(a) c_{ij} *only depends on* x_i $(i,j = 1,\dots,n)$.

(b) L_k *commutes with* L_1 *for* $k = 2,\dots,n$.

(c) *The operators* $L_1,\dots,L_n$ *mutually commute.*

PROOF. A calculation shows that

$$[L_k,L_\ell] := L_kL_\ell - L_\ell L_k = \sum_{i,j=1}^n\left[\left(\gamma_{ki}\frac{\partial\gamma_{\ell j}}{\partial x_i}-\gamma_{\ell i}\frac{\partial\gamma_{kj}}{\partial x_i}\right)\left(2\frac{\partial}{\partial x_i}+b_i(x_i)\right)+\right.$$

$$\left.+\left(\gamma_{ki}\frac{\partial^2\gamma_{\ell j}}{\partial x_i^2}-\gamma_{\ell i}\frac{\partial^2\gamma_{kj}}{\partial x_i^2}\right)\right]\left(\frac{\partial^2}{\partial x_j^2}+b_j(x_j)\frac{\partial}{\partial x_j}+c_j(x_j)\right).$$

Now apply Lemma 4.2. $\square$

The implication (a) ⇒ (c) in the above theorem formally coincides with a result in KÄLLSTRÖM & SLEEMAN [6, Theorem 1]. (See also ATKINSON [1, Theorem 6.7.2].) These authors are working in a Hilbert space context. Our result is obtained from formula (0.1) in [6] by putting $\alpha_0 := 1$, $\alpha_i := 0$ $(i = 1,\ldots,n)$, $A_i := -(\partial^2/\partial x_i^2 + b_i(x_i)\partial/\partial x_i + c_i(x_i))$, $S_{ij} := c_{ij}(x_i)$.

Let F, G be C^∞-functions in the 2n real variables $x = (x_1,\ldots,x_n)$, $p = (p_1,\ldots,p_n)$. The *Poisson bracket* of F and G is defined by

$$\{F,G\} := \sum_{k=1}^{n} \left(\frac{\partial F}{\partial x_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial x_k}\right).$$

THEOREM 4.4. *Let* c_{ij} *and* γ_{ij} *be as in Assumption* 4.1. *Consider the* n *functions*

$$F_k(x,p) := \sum_{i=1}^{n} \gamma_{ki}(x)p_i^2.$$

Then the following three statements are equivalent:

(a) c_{ij} *only depends on* x_i $(i,j = 1,\ldots,n)$.

(b) $\{F_k,F_1\} = 0$ $(k = 2,\ldots,n)$.

(c) $\{F_k,F_\ell\} = 0$ $(k,\ell = 1,\ldots,n)$.

PROOF. We have

$$\{F_k,F_\ell\} = 2\sum_{i,j=1}^{n}\left(\gamma_{\ell i}\frac{\partial\gamma_{kj}}{\partial x_i} - \gamma_{ki}\frac{\partial\gamma_{\ell j}}{\partial x_i}\right)p_i p_j^2.$$

Now apply Lemma 4.2. □

Theorem 4.4 is contained in a result by EISENHART [3, p.289], who gives necessary and sufficient conditions for the existence of orthogonal separable coordinate systems for the Hamilton-Jacobi equation on a Riemannian manifold. These conditions involve the existence of n-1 independent quadratic first integrals for the equations of geodesics. Eisenhart's conditions were considerably improved by KALNINS & MILLER [7, Theorem 6].

THEOREM 4.5. *Consider the analytic p.d.e.*

$$\text{(4.1)}\qquad \sum_{i=1}^{n} A_i(x)\left(\frac{\partial^2 u}{\partial x_i^2} + b_i(x_i)\frac{\partial u}{\partial x_i}\right) + C(x)u = 0,$$

with $A_i(x) \neq 0$ *for all* x *and* i. *Suppose* (4.1) *separates into* n *second order o.d.e.'s. Then there are* n *linearly independent, mutually commuting linear partial differential operators* $L_1,\ldots,L_n$ *of second order such that* $L_1 = \Phi L$ *for some analytic function* Φ $(\Phi(x)\neq 0)$ *and such that all solutions* u *of* (4.1) *of the form* $u(x) = \prod_{i=1}^{n} X_i(x_i)$ *are joint eigenfunctions of* $L_2,\ldots,L_n$.

PROOF. By the separability assumption, condition (iii) of Theorem 3.1 holds. Let $(c_{ij}(x_i))$ be the matrix considered there and put

$$\tilde{c}_{i1}(x_i) := \delta_{i1}, \quad \tilde{c}_{ij}(x_i) := c_{ij}(x_i), \quad i = 1,\ldots,n, \quad j = 2,\ldots,n.$$

Let $(\gamma_{ij}(x))$ be the inverse of the matrix $(\tilde{c}_{ij}(x_i))$. Then, by (3.4), $\gamma_{1i}(x) = A_i(x)/A_1(x)$. Define the operators L_k by

$$L_k := \sum_{i=1}^{n} \gamma_{ki}(x)\left(\frac{\partial^2}{\partial x_i^2} + b_i(x_i)\frac{\partial}{\partial x_i} + c_{i1}(x_i)\right), \qquad i = 1,\ldots,n.$$

Then $L_1 = A_1^{-1} L$, the operators L_k are linearly independent and, by Theorem 4.3, they mutually commute. Let $u(x) = \Pi_{i=1}^{n} X_i(x_i) \neq 0$ be a solution of (4.1). By Theorem 3.1, there is $(\alpha_2,\ldots,\alpha_n) \in \mathbb{C}^{n-1}$ such that $(X_1,\ldots,X_n)$ satisfy the o.d.e.'s (3.6). Hence

$$\frac{\partial^2 u}{\partial x_i^2} + b_i(x_i)\frac{\partial u}{\partial x_i} + (c_{i1}(x_i) + \sum_{j=2}^{n} \alpha_j c_{ij}(x_i))u = 0, \qquad i = 1,\ldots,n.$$

Multiplication of this equation by $\gamma_{ki}(x)$ $(k = 2,\ldots,n)$ and summation over i yields $L_k u + \alpha_k u = 0$. □

Theorem 4.5 is well-known for many special separable second order p.d.e.'s (cf. MILLER [9]), but it seems that the general statement of the theorem has not been proved before.

Obviously, there still exists a commuting family of operators $L_1,\ldots,L_n$ with $L_1 = \Phi L$ if $Lu = 0$ is separable after a transformation of coordinates. Conversely assume that a second order operator L can be extended to a family of n linearly independent, mutually commuting second order operators $L_1 := L, L_2,\ldots,L_n$. It would be interesting to formulate a criterium, under which conditions the existence of such a commuting set implies that L is separable into second order o.d.e.'s after some transformation of coordinates. The result by KALNINS & MILLER [7, Theorem 6], which we already mentioned, may be helpful in achieving such a criterium.

The next theorem was already proved by EISENHART [3, p.289]. We include the proof in order to make the paper more self-contained. The subsequent corollary may be new.

THEOREM 4.6. *Let $A_1,\ldots,A_n$ be analytic functions on an open connected region Ω in $\mathbb{R}^n$, not taking the value zero. Then, locally, the following two statements are equivalent:*

(a) *There are analytic functions c_{ij} $(i,j = 1,\ldots,n)$ in one real variable with $\det(c_{ij}(x_i)) \neq 0$ such that*

$$\sum_{i=1}^{n} c_{ij}(x_i)A_i(x) = \delta_{j,1}, \qquad j = 1,\ldots,n.$$

(b) *The functions A_i satisfy the system of p.d.e.'s*

(4.2) $$A_{j;p,q} = 0, \qquad p,q,j = 1,\ldots,n, \qquad p \neq q,$$

where

$$A_{j;p,q} := \frac{\partial^2 \log A_j}{\partial x_p \partial x_q} + \frac{\partial \log A_j}{\partial x_p}\frac{\partial \log A_j}{\partial x_q} - \frac{\partial \log A_p}{\partial x_q}\frac{\partial \log A_j}{\partial x_p} + - \frac{\partial \log A_q}{\partial x_p}\frac{\partial \log A_j}{\partial x_q}. \tag{4.3}$$

PROOF. First we prove (a) ⇒ (b). Suppose that (a) holds. Let $(\gamma_{ij}(x))$ be the matrix inverse of $(c_{ij}(x_i))$. Then (b) of Lemma 4.2 holds. Put

$$\rho_{kj}(x) := \gamma_{kj}(x)/\gamma_{1j}(x).$$

Then, for each $k = 1,\ldots,n$, the functions $\rho_{k1},\ldots,\rho_{kn}$ satisfy the system

$$\frac{\partial \rho_{kj}}{\partial x_p} = (\rho_{kp}-\rho_{kj}) \frac{\partial \log A_j}{\partial x_p}, \qquad j,p = 1,\ldots,n. \tag{4.4}$$

Hence

$$0 = \frac{\partial}{\partial x_p}\left(\frac{\partial \rho_{kj}}{\partial x_q}\right) - \frac{\partial}{\partial x_q}\left(\frac{\partial \rho_{kj}}{\partial x_p}\right) = (\rho_{kp}-\rho_{kq})A_{j;p,q}.$$

Fix p and q with $p \neq q$. Then $\rho_{kp} \neq \rho_{kq}$ for some k, because, otherwise, $\gamma_{kp} = (\gamma_{1p}/\gamma_{1q})\gamma_{kq}$ $(k = 1,\ldots,n)$, contradicting the nonsingularity of the matrix $(\gamma_{ij}(x))$. Thus $A_{j;p,q} = 0$ for $p \neq q$.

Next we prove (b) ⇒ (a). Suppose (4.2) holds. By a theorem of Frobenius (cf. DIEUDONNÉ [2,Ch.X, p.314]) this implies the complete integrability of the system

$$\frac{\partial \rho_j}{\partial x_p} = (\rho_p-\rho_j) \frac{\partial \log A_j}{\partial x_p}.$$

Hence, locally, there are n linearly independent analytic solutions $(\rho_{k1},\ldots,\rho_{kn})$, $k = 1,\ldots,n$, of this system, including the trivial solution $(\rho_{11},\ldots,\rho_{1n}) :=$ $:= (1,\ldots,1)$. Put $\gamma_{kj} := A_j\,\rho_{kj}$. Then (4.4) implies (b) of Lemma 4.2. Since $(\gamma_{ij}(x))$ is a nonsingular matrix, we conclude that (a) of Lemma 4.2 is valid for its matrix inverse $(c_{ij}(x_i))$. This proves (a) of the present theorem. □

COROLLARY 4.7. *Let* $A_1,\ldots,A_n$ *be analytic functions on an open connected region* Ω *in* $\mathbb{R}^n$, *not taking the value zero. Then, locally, the following two statements are equivalent:*

(a) *There are analytic functions* c_{ij} $(i = 1,\ldots,n;\ j = 2,\ldots,n)$ *in one real variable with* rank $(c_{ij}(x_i)) = n-1$ *such that*

$$\sum_{i=1}^{n} c_{ij}(x_i)A_i(x) = 0, \qquad j = 2,\ldots,n.$$

(b) *Let* $A_{j;p,q}$ *be defined by* (4.3). *Then*

$$A_{1;p,q} = A_{2;p,q} = \dots = A_{n;p,q}, \qquad p,q = 1,\dots,n, \quad p \neq q.$$

PROOF. First we prove (a) $\Rightarrow$ (b). Suppose that (a) holds. Let $c_{i1}(x_i) := \delta_{i1}$ $(i = 1,\dots,n)$. Let, $\tilde{A}_i := A_i/A_1$ $(i = 1,\dots,n)$. Then the $\tilde{A}_i$'s satisfy (a) of Theorem 4.6. Hence $\tilde{A}_{j;p,q} = 0$ $(p\neq q)$. A calculation shows that $\tilde{A}_{j;p,q} = A_{j;p,q} - A_{1;p,q}$. Hence $A_{j;p,q} = A_{1;p,q}$ $(j = 2,\dots,n;\ p \neq q)$.

Next assume (b) and put again $\tilde{A}_j := A_j/A_1$. Then

$$0 = A_{j;p,q} - A_{1;p,q} = \tilde{A}_{j;p,q} \qquad (p\neq q),$$

so (a) of Theorem 4.6 is valid for the $\tilde{A}_i$'s, hence (a) of the corollary holds for the A_i's.

5. SEPARATION OF VARIABLES AND EQUATIONS ON RIEMANNIAN MANIFOLDS

In (2.4) we introduced the Laplace-Beltrami operator

$$\Delta = \sum_{i=1}^{n} g^{-\frac{1}{2}} \frac{\partial}{\partial x_i} \circ g^{\frac{1}{2}} g_i^{-\frac{1}{2}} \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} g_i^{-1}\left(\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial x_i} \log(g^{\frac{1}{2}}/g_i)\right) \frac{\partial}{\partial x_i}\right) \tag{5.1}$$

for orthogonal local coordinates on a Riemannian manifold. More generally, we will consider operators Δ of the form (5.1) without this geometric interpretation, so it is no longer required that $g_i(x) > 0$, but g_i may be a complex-valued analytic function on an open connected region Ω in $\mathbb{R}^n$, with $g_i(x) \neq 0$ on Ω. We still put $g := \prod_{i=1}^{n} g_i$. Note that the second expression for Δ in (5.1) is independent of the choice of the branches for the square root or the logarithm.

Our first result, contained in Theorem 5.1 below, shows that a linear second order p.d.e. in $n \geq 3$ variables which separates into second order o.d.e.'s can always be written in the form $\Delta u + V(x)u = 0$, with Δ as in (5.1). Historically, a general theory of separation of variables was usually given in the context of a Riemannian manifold. Our theorem shows that this meant no loss of generality.

THEOREM 5.1. *Let*

$$L := \sum_{i=1}^{n} A_i(x)\left(\frac{\partial^2}{\partial x_i^2} + b_i(x_i)\frac{\partial}{\partial x_i}\right) \tag{5.2}$$

be a partial differential operator on an open connected region Ω *in* $\mathbb{R}^n$, *where* A_i *and* b_i *are analytic and* $A_i(x) \neq 0$ *on* Ω. *Suppose that the p.d.e.* $Lu = 0$ *separates into second order o.d.e.'s. Then for each analytic function* Φ *on* Ω $(\Phi(x) \neq 0$ *on* $\Omega)$ *there is an analytic function* R *on* Ω $(R(x) \neq 0$ *on* $\Omega)$ *such that*

$$\Phi L = R^{-1}\Delta \circ R - R^{-1}(\Delta R), \tag{5.3}$$

where Δ *is given by* (5.1) *and*

$$(5.4)\qquad g_i := \frac{1}{\Phi A_i}\,.$$

If $n \geq 3$ *then, in particular, we can choose* Φ *such that* $R \equiv 1$, *i.e.*

$$\Phi L = \Delta, \qquad g_i = 1/\Phi A_i.$$

PROOF. By the separability of $Lu = 0$, condition (iii)' of Theorem 3.1 holds. For the matrix $c_{ij}(x_i)$ introduced there, let $M_{i1}(x)$ be defined by (3.8) and let R and Φ be functions related by

$$(5.5)\qquad R^2\Phi^{1-n/2} = \exp\left(\sum_i \int_{x_{0i}}^{x_i} b(y_i)\,dy_i\right) \frac{(\prod_{i=1}^n M_{i1})^{1/n}}{(\prod_{i=1}^n A_i)^{1/n-1/2}}$$

for some $(x_{01},\ldots,x_{0n}) \in \Omega$. Let g_i be given by (5.4). Then

$$R^{-1}\Delta \circ R - R^{-1}(\Delta R) = \sum_{i=1}^n g_i^{-1}\left(\frac{\partial^2}{\partial x_i^2} + \left(\frac{\partial}{\partial x_i}\log\frac{R^2 g^{\frac12}}{g_i}\right)\frac{\partial}{\partial x_i}\right)$$

and

$$\frac{\partial}{\partial x_i}\log\frac{R^2 g^{\frac12}}{g_i} = \frac{\partial}{\partial x_i}\log\frac{R^2\Phi^{1-n/2}A_i}{(\prod_{i=1}^n A_i)^{\frac12}} = b_i(x_i) + \frac{\partial}{\partial x_i}\log\left(A_i \prod_{i=1}^n (M_{i1}/A_i)^{1/n}\right) =$$
$$= b_i(x_i) + \frac{\partial}{\partial x_i}\log M_{i1} = b_i(x_i),$$

where we used (5.4), (5.5), (3.10) and the fact that M_{i1} does not depend on x_i. Formula (5.3) now follows immediately. □

The second topic of this section deals with EISENHART's [3] condition on the vanishing of the Ricci tensor off the diagonal, which is necessary for separability of (2.5). Let Ω be an open connected region in $\mathbb{R}^n$ on which a fundamental tensor g_{ij} is defined, where the functions g_{ij} are complex-valued and analytic on Ω and $\det(g_{ij}(x)) \neq 0$. We may call Ω, together with the tensor g_{ij}, a formal Riemannian manifold. The Christoffel symbol of the second kind associated with this formal Riemannian structure, which we denote by Γ^i_{kj}, is defined by

$$2\sum_\ell \Gamma^\ell_{kj}\, g_{i\ell} = \frac{\partial}{\partial x_k} g_{ij} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_i} g_{jk}$$

(cf. EISENHART [4, (7.2)]). Next, theRiemannian curvature tensor is expressed in terms of these Christoffel symbols by

$$R^k_{\ell ij} := \sum_p (\Gamma^p_{j\ell}\Gamma^k_{ip} - \Gamma^p_{i\ell}\Gamma^k_{jp}) + \frac{\partial}{\partial x_i}\Gamma^k_{j\ell} - \frac{\partial}{\partial x_j}\Gamma^k_{i\ell}$$

(cf. EISENHART [4, (8.3)]). The Ricci tensor R_{ij} is obtained by contraction of the Riemannian curvature tensor:

$$R_{ij} := \sum_k R^k_{ijk}$$

(cf. EISENHART [4, p.21]); it is a symmetric tensor. Ω is called a (formal) *Einstein space* if

$$R_{ij}(x) = f(x)g_{ij}(x)$$

for some scalar function f. Clearly, on an Einstein space the Ricci tensor is diagonal if the fundamental tensor is diagonal. The class of Einstein spaces includes all Riemannian spaces of dimension 2, the flat spaces and the spaces of constant curvature.

If g_{ij} is diagonal and $g_i := g_{ii}$ then

$$(5.6) \qquad R_{pq} = \tfrac{1}{4} \sum_{j\neq p,q} \left[2\frac{\partial^2 \log g_j}{\partial x_p \partial x_q} + \frac{\partial \log g_j}{\partial x_p}\frac{\partial \log g_j}{\partial x_q} - \frac{\partial \log g_p}{\partial x_q}\frac{\partial \log g_j}{\partial x_p} + {} - \frac{\partial \log g_q}{\partial x_p}\frac{\partial \log g_j}{\partial x_q}\right], \quad p \neq q,$$

which follows from EISENHART [4, p.119].

<u>THEOREM 5.2</u>. *Let* $n \geq 3$. *Consider the p.d.e.*

$$(5.7) \qquad \Delta u + Cu := \sum_{i=1}^{n} \frac{1}{g_i}\left(\frac{\partial^2 u}{\partial x_i^2} + \left(\frac{\partial}{\partial x_i} \log \frac{g^{\frac{1}{2}}}{g_i}\right)\frac{\partial u}{\partial x_i}\right) + Cu = 0.$$

(a) *Suppose that condition* (iii)" *of Theorem* 3.1 *holds for* $A_i := g_i^{-1}$, $C(x) \equiv 1$. *Then* (5.7) *with* $C(x) \equiv 1$ *separates into second order o.d.e.'s if and only if the corresponding Ricci tensor is diagonal.*

(b) *If the Ricci tensor is diagonal and if* (5.7) *separates into second order o.d.e.'s for some specific function* $C = C_0$ *then* (5.7) *separates into second order o.d.e.'s for the function* $C(x) \equiv 1$.

<u>PROOF</u>. Let $A_i := g_i^{-1}$ and let $A_{j;p,q}$ be defined by (4.3). Under the assumptions of (a) we have $A_{j;p,q} = 0$ $(p\neq q)$, cf. Theorem 4.5, and the separation assumption of (b) implies $A_{j;p,q} = A_{1;p,q}$ $(p\neq q)$, cf. Theorem 3.1 and Corollary 4.7. So we have $A_{j;p,q} = A_{1;p,q}$ $(p\neq q)$ in both cases. It follows from (5.6) that, for $p \neq q$,

$$R_{pq} = \tfrac{1}{4}(n-2)A_{1;p,q} + \tfrac{3}{4}\sum_{k\neq p,q} \frac{\partial^2 \log g_k}{\partial x_p \partial x_q}.$$

Putting $b_i(x) := \dfrac{\partial \log(g^{\frac{1}{2}}/g_i)}{\partial x_i}$ we also have

$$\sum_{k\neq p,q} \frac{\partial^2 \log g_k}{\partial x_p \partial x_q} = \frac{\partial^2}{\partial x_p \partial x_q} \log \left(\frac{g}{g_p g_q}\right) = 2\, \frac{\partial^2 \log(g^{\frac{1}{2}}/g_p)}{\partial x_p \partial x_q} = 2\, \frac{\partial b_p}{\partial x_q}\,,$$

since $M_{p1}\, g_p = M_{q1}\, g_q$ (cf. (3.10)) and M_{i1} does not depend on x_i. Hence

$$R_{pq} = \tfrac{1}{4}(n-2)A_{1;p,q} + \frac{3}{2}\frac{\partial b_p}{\partial x_q}\,. \tag{5.8}$$

Formula (5.8) implies that, if two of the three expressions $R_{p,q}$, $A_{1;p,q}$ and $\partial b_p/\partial x_q$ vanish then also the third one vanishes. This yields precisely the three required implications in (a) and (b) of the theorem (use Theorems 3.1 and 4.6). □

Part (a) of Theorem 5.2 was already proved by EISENHART [3]. Part (b) may be new. It would be of interest to prove or disprove the two implications in (a) of Theorem 5.2 without the assumption that $C(x) \equiv 1$, but still assuming that $A_i := g_i^{-1}$ and C satisfy condition (iii) of Theorem 3.1.

Part (b) of Theorem 5.2 is related to ROBERTSON's [14] result that the equation

$$\Delta u + k^2(E-V(x))u = 0 \tag{5.9}$$

(Δ given by (5.1), $k \neq 0$) separates into second order o.d.e.'s simultaneously for all constants E if and only if (i) the coefficients $A_i := g_i^{-1}$ satisfy condition (iii)" of Theorem 3.1 with $C(x) \equiv 1$, (ii) $\partial \log(g^{\frac{1}{2}} g_i^{-1})/\partial x_i$ only depends on x_i and (iii) V is of the form $V(x) = \sum_i c_i(x)/g_i(x)$. Our theorem shows that, in case of a diagonal Ricci tensor (for instance, on an Einstein manifold), Robertson's conditions already hold if (5.9) separates for one specific value of E.

We conclude this paper with an example of a quite general class of separable p.d.e.'s of the form $\Delta u + \alpha u = 0$.

<u>EXAMPLE 5.3</u>. Consider a formal n-dimensional Riemannian manifold Ω with diagonal fundamental tensor

$$g_i(x) := \frac{1}{f_i(x_i)} \prod_{k\neq i} (x_i - x_k)\,,$$

where $f_i(x_i) \neq 0$ and $x_i \neq x_j$ $(i \neq j)$ for $x \in \Omega$. Then

$$\Delta = \sum_{i=1}^{n} \frac{f_i(x_i)}{\prod_{k\neq i}(x_i - x_k)} \left(\frac{\partial^2}{\partial x_i^2} + \tfrac{1}{2}\,\frac{f_i'(x)}{f_i(x_i)}\,\frac{\partial}{\partial x_i}\right).$$

It turns out that the p.d.e.

$$\Delta u + \alpha_1 u = 0 \tag{5.10}$$

separates into the o.d.e.'s

$$(5.11) \qquad X_i''(x_i) + \tfrac{1}{2}\,\frac{f_i'(x_i)}{f_i(x_i)}\,X_i'(x_i) + \frac{\sum_{j=1}^{n} \alpha_j x^{n-j}}{f_i(x_i)}\,X_i(x_i) = 0.$$

If n is small and $f_i(x_i)$ is a polynomial of low degree, the equations (5.11) yield well-known equations of mathematical physics. For instance, if n = 2 and $f_i(x) = x(1-x)$ then (5.11) becomes Mathieu's equation (cf. ERDÉLYI [5, 16.2(3)]) and (5.10) is just the two-variable Helmholtz equation in elliptic coordinates (cf. MILLER [9, p.19]), disguised in algebraic form. The case that n = 3 and f is a fourth degree polynomial arises from a certain R-separable form of the three-variable Laplace equation (cf. MILLER [9, p.209]).

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